ECE 287A: Convex Optimization and Applications **Fall 2006**

Homework 3

Lecturer: Gert Lanckriet

Due: Thursday 11/09/06 (in class)

1. Consider the following quadratic programming (QP) problem

$$\min_x \frac{1}{2}x^T P x + q^T x + r, \qquad \text{s.t.} \quad -1 \leq x_i \leq 1, \quad i = 1, 2, 3,$$

where

$$P = \begin{pmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{pmatrix}, \quad q = \begin{pmatrix} -22.0 \\ -14.5 \\ 13.0 \end{pmatrix}, \quad r = 1.$$

(a) *(3 points)* This problem can be solved in Matlab using its built-in function for QP, or using one of several toolboxes and their interfaces (e.g., but not limited to, Sedumi, Mosek, cvx). Make yourself familiar with these (or other) toolboxes and solve the QP using some of them (the class website contains links to webpages with more information, under "Software"). Which optimum x^* do you obtain? Hand in your code for two different solvers of your choice.

(b) *(2 points)* Now, prove that x^* is optimal.

2. *(2 points)* Formulate the following problem as an LP:

$$\min_x ||Ax - b||_1 + ||x||_\infty ,$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

3. *(3 points) Robust LP with interval coefficients.* Consider the following LP

$$\min_x c^T x \quad \text{s.t.} \;\; Ax \succeq b.$$

Assume now that each coefficient of A is only known to lie in an interval, i.e.,

$$A \in \mathcal{A} = \{A \in \mathbb{R}^{m \times n} \mid \overline{A}_{ij} - V_{ij} \leq A_{ij} \leq \overline{A}_{ij} + V_{ij}, \; i = 1, \ldots, m, \; j = 1, \ldots, n\},$$

with $\overline{A}$ and V given. Formulate a robust approach to solve the previous LP with uncertain coefficients. Express the robust counterpart as an LP (the LP you construct should be efficient, i.e., it should not have dimensions that grow exponentially with n or m).

4. Give an explicit solution of the following QCQPs:

   (a) *(3 points)*
   $$\min_x c^T x \quad \text{s.t.} \;\; x^T A x \leq 1,$$
   where $A \in \mathbf{S}^n_{++}$ and $c \neq 0$. What is the solution if the problem is not convex, i.e., $A \notin \mathbf{S}^n_+$?

   (b) *(3 points)*
   $$\min_x x^T B x \quad \text{s.t.} \;\; x^T A x \leq 1,$$
   where $A \in \mathbf{S}^n_{++}$ and $B \in \mathbf{S}^n_+$. Also consider the non-convex extension with $B \notin \mathbf{S}^n_+$?

5. *(3 points)* Is it possible to cast the constraint $x^T x \leq yz$, with $x \in \mathbb{R}^n$ and $y, z \in \mathbb{R}_+$, as a second-order cone constraint? Explain your answer.

6. *(2 points)* Express the problem "minimize $p(x)/(r(x) - q(x))$, subject to $r(x) > q(x)$, where p, q are posynomials and r is a monomial" as a geometric programming (GP) problem (i.e., in the format (4.43) of the textbook).

7. *(3 points)* Consider the following optimization problem
   $$\min \frac{(c^T x)^2}{d^T x}, \quad \text{s.t.} \;\; Ax \preceq b.$$
   Assume $d^T x > 0$ whenever $Ax \preceq b$. Can you formulate this non-linear optimization problem as a semidefinite programming (SDP) problem?

8. Let $A(x) = A_0 + x_1 A_1 + \ldots + x_n A_n$ where $A_i \in \mathbf{S}^m$. Let $\lambda_1(x) \geq \lambda_2(x) \geq \ldots \geq \lambda_m(x)$ denote the eigenvalues of $A(x)$. Show how to pose the following problems as SDPs.

   (a) *(2 points)* Minimize the maximum eigenvalue $\lambda_1(x)$.

   (b) *(2 points)* Minimize the spread of the eigenvalues, $\lambda_1(x) - \lambda_m(x)$.

   (c) *(3 points)* Minimize the sum of the absolute values of the eigenvalues, $|\lambda_1(x)| + |\lambda_2(x)| + \ldots + |\lambda_m(x)|$. (*Hint*: express $A(x)$ as $A(x) = A_+ - A_-$ where $A_+ \succeq 0$, $A_- \succeq 0$.)

9. Consider the optimization problem
   $$\min_x x^2 + 1, \quad \text{s.t.} \;\; (x-2)(x-4) \leq 0,$$
   with $x \in \mathbb{R}$.

   (a) *(1 point)* First, analyze the primal problem. Give the feasible set, the optimal value and the optimal solution.

(b) *(2 points)* Now, analyze the Lagrangian and the dual function. Plot the objective $x^2 + 1$ versus x and, on the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $\mathcal{L}(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^* \geq \inf_x \mathcal{L}(x, \lambda)$ for $\lambda \geq 0$). Derive and sketch the Lagrange dual function g.

(c) *(2 points)* Analyze the Lagrange dual problem: state the dual problem (verify that it is a concave maximization problem), find the dual optimal value and dual optimal solution λ^*. Does strong duality hold?

(d) *(2 points)* Finally, perform the following sensitivity analysis. Let $p^*(u)$ denote the optimal value of the problem

$$\min_x x^2 + 1, \quad \text{s.t.} \quad (x-2)(x-4) \leq u,$$

as a function of the parameter u. Plot $p^*(u)$. What is $dp^*(0)/du$?

10. We consider the portfolio risk-return trade-off problem (see page 185 in the textbook), with the following data:

$$p = \begin{pmatrix} .12 \\ .10 \\ .07 \\ .03 \end{pmatrix}, \quad \begin{pmatrix} .0064 & .0008 & -.0011 & 0 \\ .0008 & .0025 & 0 & 0 \\ -.0011 & 0 & .0004 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) *(3 points)* Solve the quadratic program

$$\min_x -\overline{p}^T x + \mu x^T \Sigma x, \quad \text{s.t.} \quad 1^T x = 1, \ x \succeq 0$$

in Matlab (using `quadprog`, `cvx`, or your favorite package), for a large number of positive values of μ (for example, 100 values logarithmically spaced between 1 and 10^7). Plot the optimal values of the expected return $\overline{p}^T x$ versus the standard deviation $(x^T \Sigma x)^{1/2}$. Also make an area plot of the optimal portfolios x versus the standard deviation (as in figure 4.12 in the textbook).

(b) *(3 points)* Assume the vector of price change p is a Gaussian random variable, with mean $\overline{p}$ and covariance Σ. Formulate the problem

$$\max \overline{p}^T x, \quad \text{s.t.} \quad \mathbf{prob}(p^T x \leq 0) \leq \eta, \ 1^T x = 1, \ x \succeq 0,$$

as a convex optimization problem, where $\eta < 1/2$ is a parameter. In this problem we maximize the expected return subject to a constraint on the probability of a negative return. Solve the problem for a large number of values of η between 10^{-4} and 10^{-1}, and plot the optimal values of $\overline{p}^T x$ versus η. Also make an area plot of the optimal portfolios x versus η.

Hint: the Matlab functions `erfc` and `erfcinv` can be used to evaluate $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2} dt$ and its inverse:

$$\Phi(u) = \frac{1}{2} \text{erfc}(-u/\sqrt{2}).$$