ECE 287A: Convex Optimization and Applications	**Fall 2006**
Homework 2	
Lecturer: Gert Lanckriet	

(please indicate whether you are taking the class for letter grade or S/U grade)

Due: Tuesday 10/24/06 (in class)

1. *(3 points)* Suppose $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$, with $b \in \mathcal{R}(A)$. Show that there exists an x satisfying
$$x \succ 0, \quad Ax = b$$
if and only if there exists no λ with
$$A^T\lambda \succeq 0, \quad A^T\lambda \neq 0, \quad b^T\lambda \leq 0.$$
This is called a *theorem of alternatives* and has several useful applications as we will see later on. (*hint*: first prove the following fact from linear algebra: $c^Tx = d$ for all x satisfying $Ax = b$ if and only if there is a vector λ such that $c = A^T\lambda, d = b^T\lambda$)

2. *(2 points)* Given are 3 concentric circles, with radius 1, 2 and 4, corresponding to the contour lines of a function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$, respectively for $f(x, y) = 1$, $f(x, y) = 2$ and $f(x, y) = 3$. Could f be convex (concave, quasiconvex, quasiconcave)? Explain your answer.

3. For each of the following functions, determine whether it is convex, concave, quasiconvex, or quasiconcave. Explain your answer.

   (a) *(2 points)* $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{++}$.

   (b) *(2 points)* $f(x) = \max_{i=1,\ldots,k} ||A^{(i)}x - b^{(i)}||$, where $A^{(i)} \in \mathbb{R}^{m\times n}, b^{(i)} \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$.

   (c) *(3 points)* $f(x) = \mathrm{tr}(A_0 + x_1A_1 + \ldots + x_nA_n)^{-1}$, on $\{x \mid A_0 + x_1A_1 + \ldots + x_nA_n \succ 0\}$, where $A_i \in S^m$.

   (d) *(2 points)* $f(x, t) = -\log(t^2 - x^Tx)$, on $\{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid t > ||x||_2\}$.

   (e) *(3 points)* $f(x, y, z) = -\sqrt{yz - x^Tx}$ on $\{(x, y, z) \mid yz > x^Tx,\ y, z > 0\}$.

   (f) *(2 points)* $f(x) = \sum_{i=1}^r |x|_{[i]}$ on $\mathbb{R}^n$, where $|x|$ denotes the vector with components $|x|_i = |x_i|$ (i.e., $|x|$ is the absolute value of x, componentwise), and $|x|_{[i]}$ is the ith largest component of $|x|$.

4. *(2 points)* In general, the product of two convex functions is not convex. However, there are some results that apply to functions on $\mathbb{R}$. Prove the following: if f and g are convex, both non-decreasing (or non-increasing), and positive functions on an interval, then the product fg is convex.

5. *(2 points)* Let x be a real-valued random variable which takes values in $\{a_1, \ldots, a_n\}$ where $a_1 < a_2 < \ldots < a_n$, with $\mathbf{Prob}(x = a_i) = p_i,\ i = 1, \ldots, n$. What can you say about the expectation and the variance of x ($\mathbf{E}(x)$ and $\mathbf{Var}(x)$), as a function of p on the probability simplex $\{p \in \mathbb{R}^n \mid p_i \geq 0, \sum_{i=1}^{n} p_i = 1\}$: is it convex, concave, quasiconvex, quasiconcave?

6. *(3 points)* Show that the sum of two log-convex functions is log-convex.

7. *(3 points)* Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex and bounded above on $\mathrm{dom} f = \mathbb{R}^n$. Show that it is constant.

8. **(a)** *(3 points)* Derive the conjugate $f^*(y)$ of the *max function* defined as $f(x) = \max_{i=1,\ldots,n} x_i$ on $\mathbb{R}^n$. What will the conjugate of $f^*(y)$ look like?

   (b) *(optional: 3 points)* Show that the conjugate of $f(X) = \mathrm{tr}(X^{-1})$ with $\mathrm{dom} f = S_{++}^n$ is

   $$f^*(Y) = -2\mathrm{tr}(-Y)^{1/2} \quad \text{with } \mathrm{dom} f^* = -S_+^n$$

   (*hint*: the gradient of f is $\nabla f(X) = -X^{-2}$).