ECE 287A: Convex Optimization and Applications **Fall 2006**

Homework 1

Lecturer: Gert Lanckriet

(please indicate whether you are taking the class for letter grade or S/U grade)

Due: Friday 10/13/06 (in class)

1. In this problem we examine the geometrical interpretation of the positive definiteness of a matrix. For each of the following cases, determine the shape of the region generated by the constraint $x^T Ax \leq 1$.

   (a) *(2 points)* $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

   (b) *(2 points)* $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

   (c) *(2 points)* $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

2. Let $A = A^T \in \mathbb{R}^{n \times n}$.

   (a) *(2 points)* Show that $\mathrm{tr}(A) = \sum_{i=1}^n \lambda_i$ with λ_i the eigenvalues of A.

   (b) *(2 points)* Show that $|A| = \det(A) = \prod_{i=1}^n \lambda_i$ with λ_i the eigenvalues of A.

   (c) *(optional: 2 bonus points)* Do the properties in (a) and (b) still hold for a general $A \in \mathbb{R}^{n \times n}$ (not necessarily symmetric)? Prove your answer.

3. Consider a partitioned matrix

$$X = X^T = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

   where A, C are square and symmetric. When C is invertible, we define $S = A - BC^{-1}B^T$ and call it the *Schur complement* of C in X. We will now show that $X \succ 0$ if and only if $C \succ 0$ and $S \succ 0$, in two different ways:

   (a) *(3 points)* Let $P = \begin{bmatrix} I & F \\ 0 & I \end{bmatrix}$ with F the same size as B above. What is the relationship between positive-definiteness of $\tilde{X} = PXP^T$ and that of X? Find F such that $\tilde{X}$ is block-diagonal, that is, $PXP^T = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$, where $*$ represents a matrix. Now, complete the proof.

(b) *(3 points)* Let $z = [u^T \ v^T]^T$. Consider the quadratic form $z^T X z$. Minimize it with respect to v (you can find information about differentiation with respect to a vector in the course reader or the matrix cookbook, posted on the website). The required result should follow immediately.

4. Show the equivalence between the following statements for $A = A^T \in \mathbb{R}^{n\times n}$:

   (a) $A \succ 0$.

   (b) *(2 points)* The eigenvalues of A are positive.

   (c) *(2 points)* If A_i represents the submatrix of A formed by taking the first i rows and i columns of A, then $\det(A_i) > 0$ for $i = 1, \ldots, n$. (hint: use induction)

5. In this problem, we will prove a result that will be used later during the course. For any pair of symmetric positive semidefinite matrices of the same size A, B, $\mathrm{tr}(AB) \geq 0$. To do this, we will proceed as follows:

   (a) *(2 points)* Let $B = U^T \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} U$, where U is unitary, $\Lambda \in \mathbb{R}^{r\times r}$ is a diagonal matrix and r is the rank of B. Also, define: $\tilde{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} = UAU^T$. Show that $\tilde{A} \succeq 0$ and that the diagonal elements of A_{11} ($a_{ii}, i = 1, \ldots, r$) are non-negative.

   (b) *(2 points)* Show that $\mathrm{tr}(AB) = \mathrm{tr}(A_{11}\Lambda)$.

   (c) *(2 points)* Conclude that $\mathrm{tr}(AB) \geq 0$.

   (d) *(optional: 2 bonus points)* What can you say if $\mathrm{tr}(AB) = 0$?

6. *(2 points)* Assume A is an orthogonal matrix. What do we know about its determinant?

7. Which of the following sets are convex? If they are affine, cones, balls, ellipsoids or polyhedra, say so. Explain your answer.

   (a) *(1 point)* A set of the form $\{x \in \mathbb{R}^n \mid \alpha \leq a^T x \leq \beta\}$.

   (b) *(1 points)* A set of the form $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, \ldots, n\}$.

   (c) *(1 points)* A set of the form $\{x \in \mathbb{R}^n \mid a_1^T x \leq b_1, a_2^T x \leq b_2\}$.

   (d) *(2 points)* The set of points closer to a given point than a given set, i.e., $\{x \mid ||x - x_0||_2 \leq ||x - y||_2, \forall y \in S\}$ where $S \subseteq \mathbb{R}^n$.

   (e) *(2 points)* The set of points closer to one set than another, i.e., $\{x \mid \mathrm{dist}(x, S) \leq \mathrm{dist}(x, T)\}$ where $S, T \subseteq \mathbb{R}^n$ and $\mathrm{dist}(x, S) = \inf\{||x - z|| \mid z \in S\}$.

   (f) *(2 points)* The set $\{a \in \mathbb{R}^k \mid p(0) = 1, |p(t)| \leq 1 \text{ for } \alpha \leq t \leq \beta\}$ where $p(t) = a_1 + a_2 t + \ldots + a_k t^{k-1}$.

(g) *(2 points)* The set $\{x \in \mathbb{R}^n \mid ||x - a||_\infty \leq 1\}$.

(h) *(2 points)* The set of positive semidefinite matrices of dimension $n \times n$.

8. Which of the following sets S are polyhedra? Why (not)? If possible, express S in the form $S = \{x \mid Ax \preceq b,\ Fx = g\}$.

   (a) *(2 points)* $S = \{x \in \mathbb{R}^n | x \succeq 0, x^T y \leq 1 \text{ for all } y \text{ with } ||y||_2 = 1\}$.

   (b) *(2 points)* $S = \{x \in \mathbb{R}^n | x \succeq 0, x^T y \leq 1 \text{ for all } y \text{ with } ||y||_1 = 1\}$.

9. *(2 points)* Show that a set is convex if and only if its intersection with any line is convex.

10. *(3 points)* Consider the set of rank-k *outer products*, defined as $\{XX^T \mid X \in \mathbb{R}^{n \times k}, \text{rank}(X) = k\}$. Describe its conic hull in simple terms.